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Mergers, Acquisitions, and Innovation

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Abstract:

Evidence shows that the merger firms are more successful in R&D than those that are not. Then how many firms are to be invested by the merger firms trying to acquire innovation? We derive a unique and closed-formed firm-level profit maximizing number of start-up entrepreneurs that the merger firm takes equity positions in trying to acquire innovation. The model is mainly described by two stochastic differential equations. In the model, we apply the Bayesian inferences to the construction of expectation on R&D process and to the rise of profit excluding acquisition-related costs caused by marketing for each stochastic differential equation.

Keywords: open innovation, bayesian inference, stochastic differential equations

JEL codes: G11, G34, O32

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Mergers, Acquisitions, and Innovation

Hidenobu Hirata/ Tadashi Yagi

> [T]here is little justification for monopoly in a world of Open Innovation.
> —Chesbrough (2003, p.194)

Roughly speaking, *Open Innovation* is a paradigm that generates ideas by several firms or acquires innovation through acquisition or capital investment. Chesbrough (2003) insists that innovation performed by only one firm is inefficient. Inventions born by the ideas that researchers serendipitously strike can not be known to fit with their firms' product beforehand. Rather there is much lower probability that serendipitous ideas that become inventions fit to their firm's product than not, in an economy where there is a broad range of products. Evidence by Griliches (1998) estimates R&D and productivity at the firm level and it unfolds that the merger firms' R&D investments are more successful than those that are not. Griliches (1998, chap. 5, pp.113-117) wrote that

> About one firm out of five in our "complete" sample … appeared to be affected … by considerable and generally simultaneous "jumps" … in gross plant, number of employees, and sales. We have been able to check and convince ourselves that most of these jumps do, in fact, result from mergers, although some may be the result of very rapid growth…. One way of dealing with this problem is simply to drop the offending firms. This results in what we have called the "restricted" sample…. The within estimates are … very sensitive, and the estimated γ collapses, declining from .11 to .05 and -.03 in the complete, intermediate, and restricted samples, respectively [where γ is the elasticity of output with respect to R&D capital] … It is clear … that the merger firms are responsible for the difference…. In other words, R&D seems most effective for firms growing rapidly through mergers, and both phenomena (mergers and R&D growth) are apparently related…. Such a finding raises questions that deserve additional analysis: Who are these "merger" firms and why would their R&D investment be more successful? What kind of selectivity is at work here? [The phrase within brackets added]

Our motivation of this paper is to derive answers to these Griliches' questions. The purpose of this paper is to solve an optimal number of start-up entrepreneurs that the merger firm takes equity positions in so as to maximize the merger firm's profits trying to acquire innovation within a specified period of time. In this paper the merger firm is the firm that is going to merger a start-up entrepreneur or the firm itself after the merger. An equity position is an equity investment made by the merger firm for the purpose of acquiring 50 percent or more of the shares issued by the start-up entrepreneurs after their IPO, the success of their R&D projects, and thus, innovation. We assume that the merger firm uses equity positions to finance them obtained by the payment that equals the value of call-option price multiplied by more than 50 percent of the shares issued per start-up entrepreneur if the start-up entrepreneur has been published.

In our model, we consider the cases where only the merger firms and the start-up entrepreneurs are involved. Because the start-up entrepreneurs are relatively small (before the IPO), the shareholders of the start-up entrepreneurs are only the merger firm or the owners of the start-up entrepreneurs. Hence, shareholder disapprovals do not happen once the merger announcement (contract) has been made for there is no one who opposes to the merger deal. And because the start-up entrepreneurs are relatively much smaller than the merger firms, regulatory considerations such as anti-monopoly do not occur.

Models that analyze acquisition and innovation have begun by Aghion and Tirole (1994a and 1994b) using the framework of Grossman and Hart (1986). It is second-best to purchase the other firm when the firm's investment is relatively larger than the other's (Grossman and Hart (1986)). Aghion and Tirole (1994a and 1994b) study the integrated case and the nonintegrated case: the integrated case is the case the customer owns the research unit. In the integrated case the customer owns and freely uses the innovation made by the research unit. We analyze the integrated case when there are several start-up entrepreneurs (research units) to finance prior to the success of R&D. The merger firm (the customer) freely uses the innovation made by the research unit after the former owns the latter. Aghion and Howitt (1996) give more complex considerations to Aghion and Tirole (1994a and 1994b) by rent-sharing between researchers and developers. By introducing the concept of researchers and developers (in their model, researchers becoming start-up entrepreneurs, hire developers for applied research that is necessary for the product to be sold), they study the positive effect of competition on growth. Klepper (1996) considers how entry, exit, market structure, and innovation – the ratio of product to process innovation – vary from the birth of industries through maturity. Arguing that firm innovation and firm growth, entry, exit , and size distribution deserve an integrated treatment, Klette and Kortum (2004) capture and analyze in their model heterogeneous firms, simultaneous exit and entry, optimal investments in expansion, explicit individual firm dynamics, and a steady-state firm size distribution. Loury (1979) investigates the relationship between the market structure and innovation through a model of non-cooperative game. He analyzes that the shape of relationship between the aggregate success rate of rivals and firm's optimal investment in R&D is inverted U. Lee and Wilde (1980), using a model similar to Loury's (1979) which assumes the reward to be the first to introduce the new technology is a fixed sum, where in Loury (1979) that is a flow, study the positive relationship between the aggregate success rate of rivals and firm's optimal investment in R&D.

Several papers, including Cowan (2002), insist that the relationship between the number of firms and the total industry R&D is an inverted-U shape. Using the model that is a discrete version of Loury's (1979), Cowan (2002) suggests that an increase in the number of firms in an industry decreases the number of R&D projects undertaken by per firm. This causes a decline in knowledge that is brought about by those R&D projects. Through this fact, he shows that the relationship between the number of firms in an industry and the rate of technological development – the total number of R&D projects undertaken in an industry – is an inverted-U shape. Aghion et al. (2005) argue that the relationship between product market competition (PMC) and innovation is an inverted-U shape, implying that there exists an optimal competition for the greatest innovation. In this model, they use "escaping competition" as an incentive for innovation.

Both Cowan (2002) and Aghion et al. (2005) assume that the total number of R&D projects M is not given and that it is a function of total number of firms in an industry. In this paper, we assume that M is constant in the short term regardless of whether or not the number of firms changes. This is because there is an upper bound to the number of researchers that firms contract with. Firms that attempt to innovate, contract with only energetic researchers who are growing. These researchers face time constraints. The time constraints of researchers and the upper bound of number of researchers restrict M. We assume that these requirements of firms for researchers do not change in the short term. Therefore, we assume that M is constant in the short term. Later, we refer to this in detail and in Section II, we refer to the longer term when there is the possibility of depressions M varies.

Section I presents our model that is mainly composed of the two stochastic differential equations. One is the incremental process of number of times of successful R&D projects with a Bayesian drift founded on the Poisson distributions. The other is the incremental process of profit excluding acquisition-related costs through marketing activities in which consumers' characteristics are perfectly known. In this section, we use the Poisson arrival rate of an innovation which depends on the researcher's success of current and past R&D. The optimal number of start-up entrepreneurs that the merger firm takes equity positions in for profit maximization is also derived in this section. Section II simulates the derived optimal number of firms that the merger firm takes equity positions in. This section also deals with a case when consumers' characteristics are imperfectly known. With its adjusted stochastic differential equation depicting incremental process of profit excluding acquisition-related costs when consumers' characteristics are imperfectly known, this section simulates the optimal number of start-up entrepreneurs that the merger firm takes equity positions in and its expected profits by those equity positions. Section III presents a brief review of our results and the conclusion.

I. The Model

A. Assumptions

For companies like Intel and Hewlett-Packard, it makes perfect sense to invest substantially in options, such as equity investments in small entrepreneurial companies with interesting technology (MacMillan and McGrath (2002)).

In this paper, the merger firm takes equity positions in n start-up entrepreneurs trying to acquire innovation. The amount c of equity investment invested for the each start-up entrepreneur i have the value of call-option price multiplied by more than 50 percent of the shares issued per start-up entrepreneur if the start-up entrepreneur has been published. Thus after the publication of the start-up entrepreneurs, the merger firm has the equal footings as the rights to buy the start-up entrepreneurs by the "exercise price" or "striking price" of call options. Profit excluding acquisition-related costs $x(t)$ and the number of times of successful R&D $j(t)$ in time t are given by stochastic differential equations, where acquisition-related costs are $nc+C$ and C have the value of call-option strike price multiplied by more than 50 percent of the shares issued per start-up entrepreneur if the start-up entrepreneur has been published. The merger firm acquires only one start-up entrepreneur in n that has the best innovation. As in Loury (1979), there are no externalities in the R&D process (no stealing of trade secrets for example). As noted before, M is constant in the short term. Firms are risk neutral. Demand varies stochastically. The merger firms are in the monopolistically competitive product markets under complete information. They can raise as much fund as they need from the financial sector at the constant short-term interest rate r. In the financial sector, complete stock markets with the Black-Scholes (1973) economy also prevail. Thus, there are no budget constraints. The stock price/earnings ratio (PER) is constant, where earnings are profits. There are no gains or losses from the acquisition itself owing to the no-arbitrage condition. Thus, we can focus our study on the profits resulted from innovation and not from the arbitrage of the acquisition. The gained assets including intangible and tangible assets (net assets) are arbitrage free owing to the complete stock markets. Hence, acquisition-related costs have no effects on the stock price $S(t)$ of the merger firm.

If the merger firm has the rights to buy the stocks of start-up entrepreneurs that have succeeded in their R&D projects prior to the success by call options, it can make the risk of investments as small as possible by setting the “exercise price” or “striking price” beforehand. Buying start-up entrepreneurs that have succeeded in R&D projects by exercising call options and not buying those that did not, the merger firm can hold down the costs of innovation.

B. Number of Times of Successful R&D Projects

In this section, we present a Bayesian drift derived from Bayesian equation founded on the Poisson distributions to express the increase in $j(t)$. The success rate of R&D projects depends on the increase of $j(t)$ by his/her past experiences in them. The researcher’s past experiences of success in R&D have positive effects on $j(t)$. Hence, they increase the success rate of R&D projects. We use the Poisson arrival rate of an innovation. The Bayesian equation founded on the Poisson distributions is

$$\lambda'' = \frac{j''}{m''} = \frac{j + j'}{m + m'} \qquad (1)$$

(Pratt, Raiffa, and Schlaifer (1995, pp.345-362)),[1] where λ'' is the posterior distribution, λ is the Poisson arrival rate of an innovation, j' and m' are the prior distributions, and $m=\frac{M}{N}=\frac{M}{n+k}$, where m is the number of times of R&D projects that are evenly allocated for each t and i by the merger firm, $N=n+k$ is the total number of firms in an industry, and k is the number of other firms that the merger firm did not take equity positions in including the number of rivals to the merger firm. In other words, k implies the number of firms that are rivals to the merger firm plus the number of the start-up entrepreneurs that the merger firm has no technological interest in. Following the update in the Bayesian equation founded on the normal distributions (Chamley (2004, p.25)), the update of j and m by the prior distributions will be

$$\begin{aligned} j''' &= j'' + j \\ &= j' + j + j \\ &= 2j + j', \end{aligned} \tag{2}$$

where j''' is the posterior distribution after updated 2 times. Assuming that $j' = j$,

$$j''' = 3j. \tag{3}$$

The same applies to m. Hence, the updated Bayesian equation founded on the Poisson distributions is

$$\begin{aligned} \lambda^{(t)} &= \frac{tj}{tm} \\ &= \frac{j}{m}. \end{aligned} \tag{4}$$

Because, m is allocated evenly for each unit of time, the increase of the success rate of R&D projects depends on j. To express the positive effects of past experiences of success in R&D on the number of times he/she succeeds, we assume that the incremental process of number of times of successful R&D is given by the stochastic differential equation[2]

$$dj(t) = \mu j(t)dt + j(t)\sqrt{\frac{j(t)}{n}}dV(t), \quad \mu \geq g. \tag{5}$$

In (5), μ is the drift of $j(t)$, the know-how, the experience how to succeed in R&D gained by the researchers and $V(t)$ depicts the Brownian motion of $j(t)$. We use the Brownian motion for the researchers' efforts are the ones that are continuous, engaging in m R&D projects in each time. Though the successful R&D seems to be discontinuous process, for the researchers themselves it is a thing that can be reached through numerous experiments including the ones that fail. Numerous experiments have the role as the path to the success. Thus we assume that $j(t)$ has a continuous process and apply Brownian motion. (5) is expressing the sequence of the number of times of R&D projects that researchers succeeded. Firms have requirements for the researchers to undertake R&D so they only contract with the researchers with $\mu \geq g$, where a constant g in the short term defines the precinct (the lower bound) of μ. Firms only contract with the researchers who are within their precincts that reflect their demands for the qualifications of researchers. When in depressions in the longer term, as firms become severe searching for researchers, their precincts become narrower, that is, g becomes larger. Thus when in depressions or when business turned from good to bad in the longer term, M becomes lower. Therefore, M is a negative function of g in the long term. Notice that in the short term, M is constant as $\overline{M}$. Firm's g does not vary in the short term. The difference between the short term and the long term, in this paper, is whether it is shorter or longer than a specified period of time $[0,T]$, that is, the term $[0,t]$ is

$$\begin{aligned} [0,t] &= \text{short term}, \quad t \leq T \\ &= \text{long term}, \quad t > T, \end{aligned} \tag{6}$$

where T is the maturity date. Even though when there are many researchers, there are upper bounds to the numbers of researchers that the firms contract with in the short term for they do not let any researchers with $\mu < g$ undertake any R&D projects. And these researchers that suffice firms' requirements face time constraints. The number of researchers that firms contract with and researchers' time constraints restrict M. We also assume that the economy is using all these resources, that is, the researchers' labor markets clear. Thus, $M = \overline{M}$ in the short term. And μ does not depend on n. The start-up entrepreneurs that the merger firm took equity positions in do not share their researchers' know-how (μ) with each other because they are rivals in terms of receiving the reward (C) from the merger firm after the success of R&D. Each start-up entrepreneur that receive equity finance from the merger firm does not have any incentive to share the information about its μ with each other. Thus, μ is not a function of n.

Discoveries made by the researchers are actual facts thus they vary stochastically. This is expressed by $\sqrt{\frac{j(t)}{n}}dV(t)$, where the variance of $j(t)$ is $j(t)$ considering the variance of the Poisson distribution $\lambda = \frac{j(t)}{m}$ and m allocated evenly for each t. The variance $j(t)$ is divided by n since the rational merger firm does not allow any of the same experiments to be undertaken by any start-up entrepreneurs that it took equity positions in. The same experiments must not be undertaken again in the present or in the future by the contracts and by the detailed plans of experiments. Reports of those are to be submitted to the merger firm after the experiments so that the start-up entrepreneurs that are receiving equity finance from the same merger firm are unable to undertake any of the same experiments. $\sqrt{\frac{j(t)}{n}}$ is the volatility.

In the next section, we present a Bayesian drift of $x(t)$ derived from the inverse of Bayesian equation founded on the Normal distributions.

C. Marketing

We assume that firms perform marketing for the new product depending on the invention. In this paper, this marketing contributes to the largest $x(t)$ through the *four Ps* (Kotler (1999, pp.32-33)):

·*Product*: The market offering itself, specifically a tangible product, packaging, and a set of services that the buyer would acquire through the purchase

·*Price*: The price of the product along with other charges that are made for delivery, warranty, and so on

·*Place (or distribution)*: The arrangements to make the product readily available and accessible to the target market

·*Promotion*: The communication activities, such as advertising, sales promotion, direct mail, and publicity to inform, persuade, or remind the target market about the product's availability and benefits

In this model, demand varies stochastically. Firms have to search for the optimal price aimed at the maximum sales which is depending on the price elasticity of demand. Assume that the increment of inverse of the Bayesian inference founded on the normal distributions per time to be the drift of $x(t)$. This is expressing the augmentation of precision per time about the true value of the four Ps that brings the largest $x(t)$ by marketing activities. Then we get the incremental process of $x(t)$ as[3]

$$dx(t) = \frac{\tilde{m}n}{\sigma^2} x(t)dt + x(t)\frac{\sigma}{\sqrt{n}} d\tilde{V}(t), \tag{7}$$

where $\tilde{m}$ is the number of times that marketing activities are performed in a time and $\tilde{V}(t)$ is the Brownian motion of $x(t)$. The second term in the RHS of (7) is derived by assuming that the merger firm disperses variance by investing equally in each start-up entrepreneur that it takes equity positions in. $x(t)$ is an actual event that is subject to fluctuation, hence this is expressed by $\frac{\sigma}{\sqrt{n}} d\tilde{V}(t)$. $\frac{\sigma}{\sqrt{n}}$ is the volatility. Further, $V(t)$ and $\tilde{V}(t)$ are independent of each other, since the former Brownian motion represents the fluctuation of $j(t)$ and the latter, the fluctuation of $x(t)$. (7) is expressing the increment of $x(t)$ whose drift is depicted by the augmentation of precision per time about the true value of the four Ps that brings the largest $x(t)$.

In this paper, $\tilde{m}$ and m are assumed to be proportional to each other. This is because one of the functions of marketing is to determine the directions of technologies before it is too late to change the characteristics of new technologies. The more experiments are conducted, the more difficult it becomes to change the directions of new technologies. Several authors insist the importance of communication between marketing and R&D departments. "[C]ompanies serious about competing via innovation must recognize that the educational task is multidirectional; urging R&D to educate marketing but not vice versa is a mistake" (Hulbert, Capon, and Piercy (2003, p.220)). Further, "[w]ithout close communication between marketing and R&D, the successful new brand development rate will be even lower than the pitiful national average of 10 to 30 percent" (Luther (2001, p.81)). Thus, we assume that an increase in m increases $\tilde{m}$. Therefore m and $\tilde{m}$ are conducted at the same pace. While deriving the optimization, we assume that $m = \tilde{m}$ due to the above mentioned reasons and simplicity of calculations. We do not discuss the allocation of resources between marketing and R&D for this problem deviates from our thesis and differs by the industries. But some discussions are made in the simulations.

D. Expected Profits of the Merger Firm

From (4), the expected profits of the merger firm π becomes[4]

$$\pi = n\frac{\bar{j}(t)}{m}x(t) - nc - C\,, \tag{8}$$

where $\bar{j} = \frac{1}{n}\sum_{i=1}^{n} j_i$. We assume that the start-up entrepreneurs receiving equity finance by selling c to the merger firm cover the cost of m and $\tilde{m}$ by c.

Before solving the merger firm's profit maximization of this problem, we state the following propositions 1 and 2.

PROPOSITION 1: *m, $\tilde{m}$, and n have no effects on* $f(t, \bar{j}(t), x(t)) = \log\{\frac{S(t)}{S(0)}\}$, where $f(t, \bar{j}(t), x(t))$ is the derivative price.

Proof: The proof of proposition 1 is given in the Appendix.

The intuition of this proposition is that as soon as m, $\tilde{m}$, *and* n are decided, the effects on $S(t)$ is formulated and thus also on $S(0)$ so as to the purchase of assets are arbitrage free.

PROPOSITION 2: *The expected production function of the merger firm with respect to n is*

$$E[n\frac{\bar{j}(t)}{m}x(t)] = x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2})t\}\,, \tag{9}$$

and is S-shaped when $Mt > 4\sigma^2$.

Proof: The proof of proposition 2 is simple calculations, after using proposition 1 and lemma 1 in the Appendix.

Proposition 2 depicts that initially, the merger firm hastens to take equity positions in other start-up entrepreneurs trying to acquire innovation. However, it is optimal for them to slow down their speed later.

Using proposition 2, we can derive the optimal number of start-up entrepreneurs that the merger firm takes equity positions in when $t = T$ as[5]

$$n^* = -k - \frac{MkT}{2\sigma^2 W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}})}, \tag{10}$$

where $\bar{\mu}$ is the drift of $\bar{j}(t)$ and W is the *Lambert W function*.[6]

The merger firm takes equity positions in the start-up entrepreneurs for innovation when $n^* > 1$ under the assumption that it allows only one section to undertake internal R&D projects that have passed a strict screening as its external R&D projects have. When $n^* = 1$, a firm performs innovation by itself. If the merger firm does not conduct any internal R&D projects, it takes equity positions in the start-up entrepreneurs for innovation when $n^* \geq 1$. When $n^* \leq 0$, the firm does not conduct any R&D projects, and thus does not exercise any call options.

If the merger firm augments n^* a unit, then N increases by a unit. This is when the merger firm helps academic researchers or those who are potential ones with regard to their start-ups. The merger firm supports those who have technologies that suit its products.

In (10), k is the number of firms that the merger firm has no technological interest in plus the number of its rivals. On account of monopolistic competition, the merger firm disregards the behavior of other firms and thus the movement of k. Therefore n^* is unique.

II. Simulations

We provide numerical simulations of n^* below. We use Maple for these simulations.

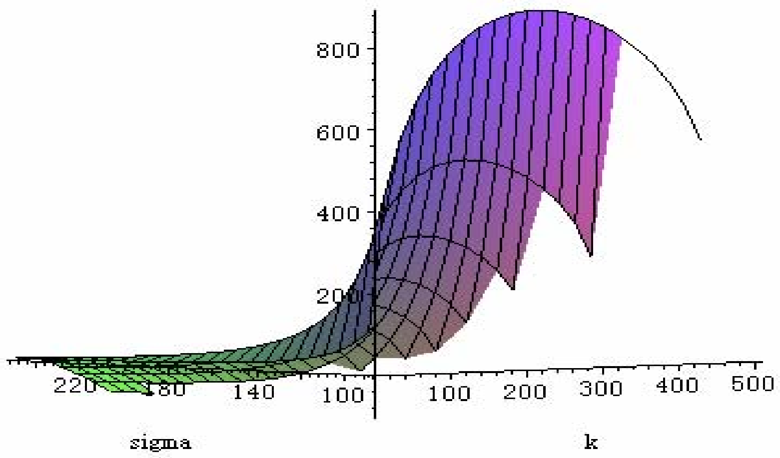

Figure 1. Volatility of the profit excluding acquisition-related costs of the merger firm, the number of firms that are rivals to the merger firm plus the start-up entrepreneurs that the merger firm has no technological interest in, and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The horizontal axes show σ and k. This figure indicates n^* on the vertical axis. σ is in the range of 90 to 250 in this figure, where c is normalized to 1, $M = 10000$, $x(0) = 100$, $\overline{\mu} = 0.01$, $r = 0.1$, and $T = 3$ are substituted.

Figure 1 is in cases when there is no observation noise on consumer behavior. From the following subsection, we model and simulate when there is an observation noise.

A. Marketing with an Observation Noise

When consumers' characteristics are not perfectly known or marketing observe a noise on consumer behavior whose characteristics are perfectly known, the Bayesian update founded on the normal distributions that is applied in (7) needs modification. Using the words of Chamley (2004, p.48), imperfect information on consumer behavior is operationally equivalent to a noise on the observation of consumer behavior whose characteristics are perfectly known. Hereafter, we modify the update of the Bayesian equation as in Chamley (2004, pp.48-50, (3.10)) so as to cope with the situations when consumer behavior is not known. The precision of marketing activities by the four Ps on $x(t)$ becomes different, and we remodel the stochastic differential equation expressing incremental process of $x(t)$ as[7]

$$dx(t) = \frac{\tilde{m}n}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2} x(t)dt + x(t)\sqrt{\frac{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2}{n}} d\tilde{V}(t), \qquad (11)$$

where ρ is the inverse of variance of the prior distribution σ_θ^2 and σ_η^2 is the variance of an observation noise. Proposition 1 can also be applied to the production function using (11) and similar proposition as proposition 2 can be derived as follows.

PROPOSITION 3: *When consumer behavior is not known, the expected production function of the merger firm with respect to* n *is*

$$E[n\frac{\bar{j}(t)}{m} x(t)] = x(0)\exp\{(\bar{\mu} + \frac{mn}{\{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2\}})t\},$$

and is S-shaped when $Mt > 4\{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2\}$.

Proof: The proof of Proposition 3 is in the Appendix.

Derivation of n^*, when there is an observation noise, is similar to the derivation of (10). And n^* becomes,

$$n^* = -k - \frac{MkT\sigma_\theta^4}{2(\sigma^2\sigma_\theta^4 + \sigma_\eta^2\sigma_\theta^4 + 2\sigma_\eta^2\sigma_\theta^2\sigma^2 + \sigma_\eta^2\sigma^4) W\left(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{M\omega\sigma_\theta^4 kT}(\sigma^2\sigma_\theta^4 + \sigma_\eta^2\sigma_\theta^4 + 2\sigma_\eta^2\sigma_\theta^2\sigma^2 + \sigma_\eta^2\sigma^4)}}\right)}. \qquad (12)$$

Hereafter, we operate our simulations based on (11), where c is normalized to 1, $M = 10000$, $x(0) = 100$, $k = 100$, $\bar{\mu} = 0.01$, $r = 0.1$, $T = 3$, $\sigma = 80$, $\sigma_{\eta} = 42.3$, $\sigma_{\theta} = 100$, and $C = 50$ are substituted except those arguments that are presented in the figures.

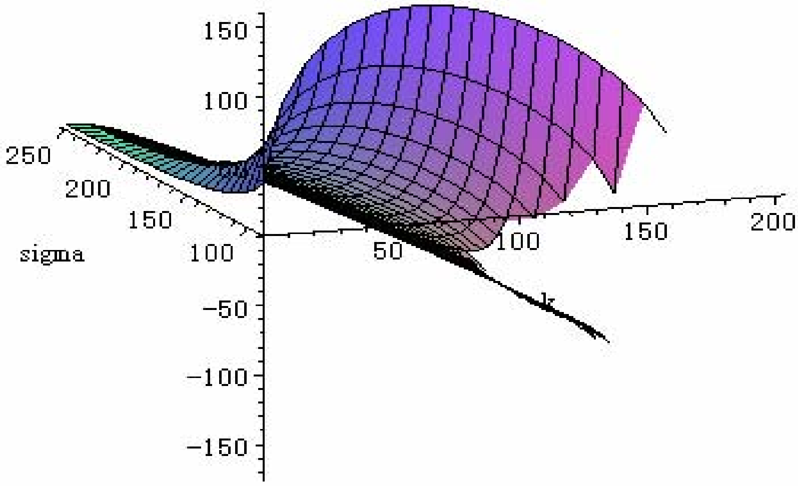

Figure 2. Version of figure 1 when there is an observation noise on consumer behavior.

In Figure 2, σ lies in the range of 90 to 250. n^{*} is much smaller when marketing observe a noise than when marketing observe no noise. Compare figures 1 and 2.

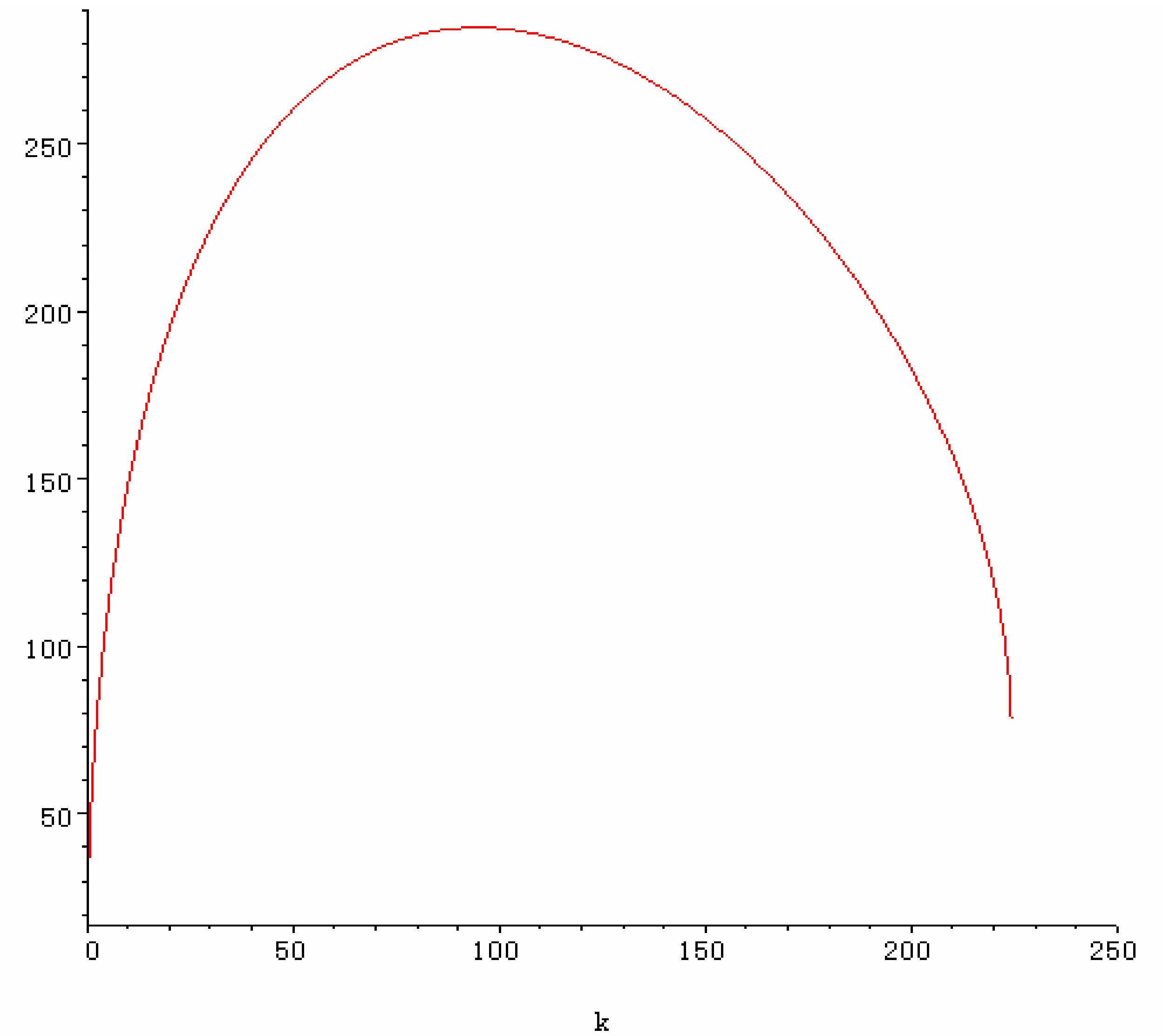

Figure 3. Number of firms that are rivals to the merger firm plus the start-up entrepreneurs that the merger firm has no technological interest in and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. Figure 3 indicates k on the horizontal axis and n^* on the vertical axis.

In figure 3, recognizing that the merger firm attempts to innovate significantly as the k begins to increase, we can see that an incentive to "escape competition" (Aghion et al. 2005) is depicted. However, when there are excessive rivals in an industry, the merger firm eventually performs no R&D or takes equity positions in no start-up entrepreneurs. This is owing to the "business stealing" effect (Aghion et al. 2005; Aghion and Howitt 1992; 1998). Using the words of Aghion and Howitt (1992), rivals do not internalize the loss to the incumbent (the merger firm) by their entry. Considering the loss in the future evoked by the entry of rivals, the merger firm decreases n^*.

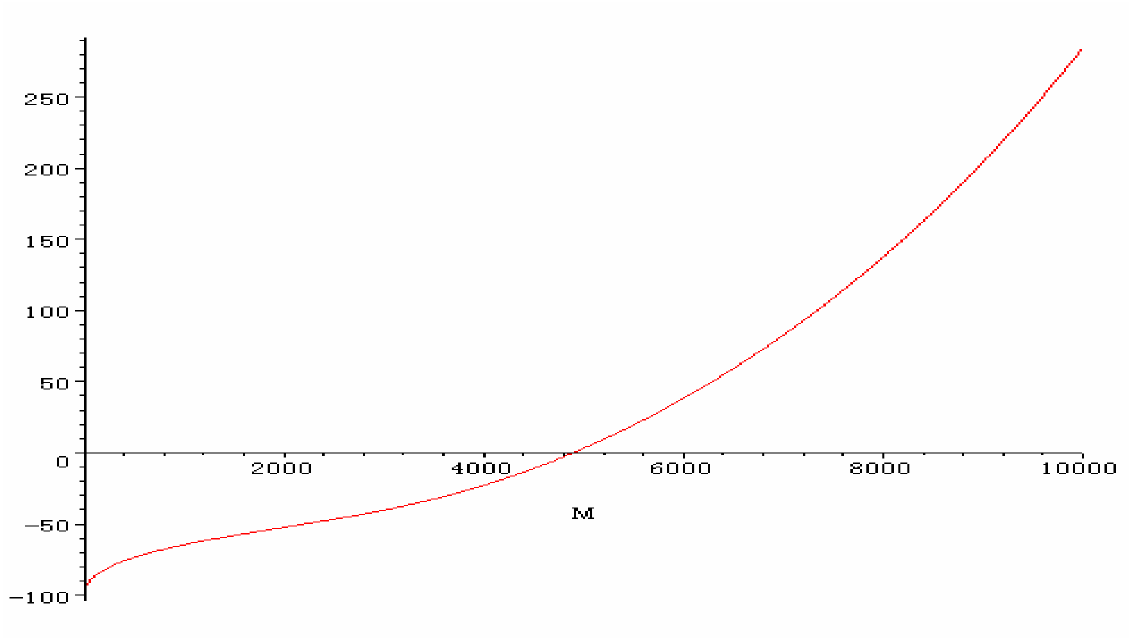

Figure 4. Total industry R&D and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. Figure 4 shows M on the horizontal axis and n^* on the vertical axis. This figure shows the transition in the long term.

Figure 4 shows the transition in the long term from no innovation to the merger firm taking equity positions in the start-up entrepreneurs trying to acquire innovation. When the number of researchers in an industry is limited and M is low, because of depression or of scant of researchers in the field, the merger firm does not perform R&D projects or does not take equity positions in other start-up entrepreneurs. This figure shows that in order to take equity positions in other start-up entrepreneurs trying to acquire innovation, the merger firm needs larger M and therefore abundant researchers engaging in R&D.

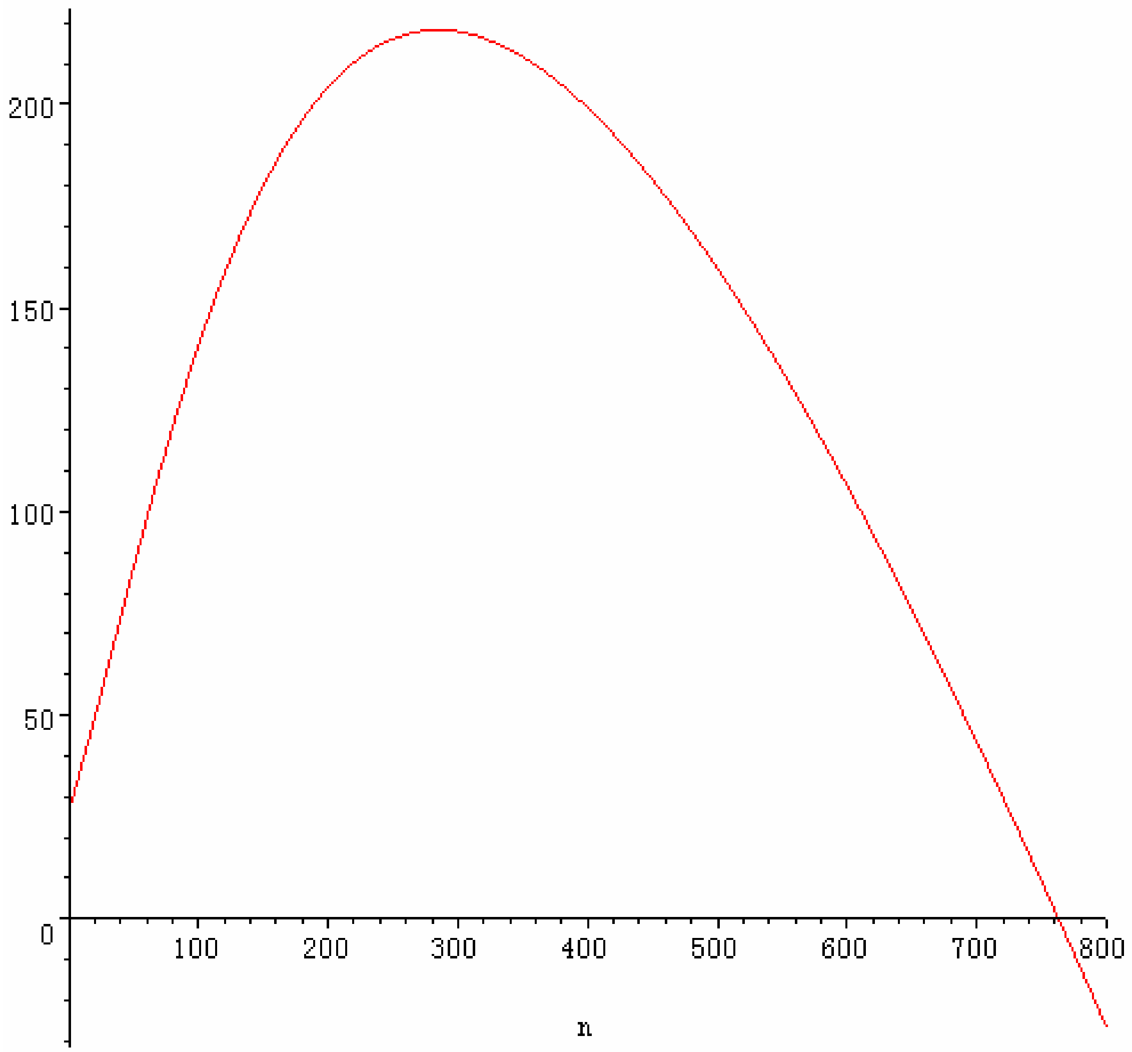

Figure 5. Number of the start-up entrepreneurs to take equity positions in by the merger firm and expected profits of the merger firm by taking equity positions in start-up entrepreneurs trying to acquire innovation. Figure 5 shows n on the horizontal axis and π of the merger firm on the vertical axis. The figure unveils that the merger firm's performance is much beyond than those that are not.

Figure 5 reveals that, according to the effective R&D projects and augmented $x(t)$ by marketing activities, the merger firm's profits is about 7 times larger at the optimal point than those that are not. We can see that the difference of the profits between the merger firms and those that are not is "very sensitive" as Griliches (1998) wrote.

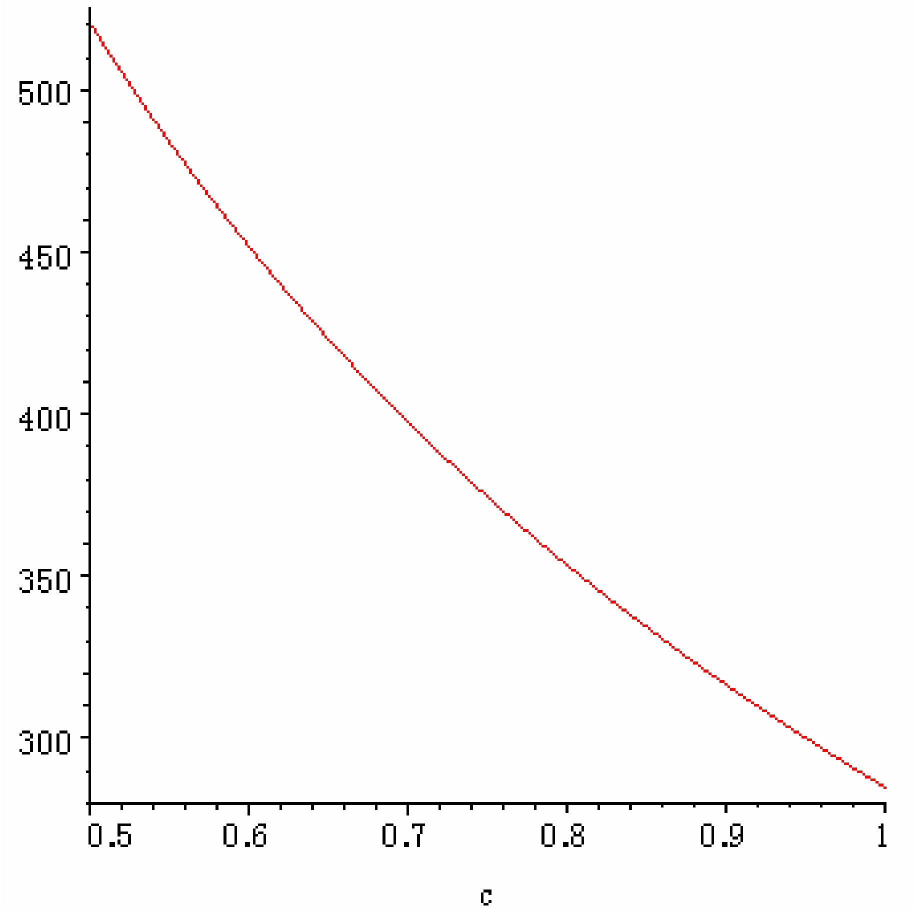

Figure 6. Merger firm's equity investments for the each start-up entrepreneur which have the value of call-option price multiplied by more than 50 percent of the shares issued per start-up entrepreneur and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. Figure 6 shows c on the horizontal axis and n^* on the vertical axis.

When costs of R&D projects and marketing activities are about the same and the merger firm pays costs of marketing on behalf of the start-up entrepreneurs, c is about 0.5. We do not discuss the optimal allocation of c between marketing and R&D in this paper, but because we are normalizing as $c = 1$, we can easily modify our model to the specific cases.

We paste the Figures 7 to 11 below.

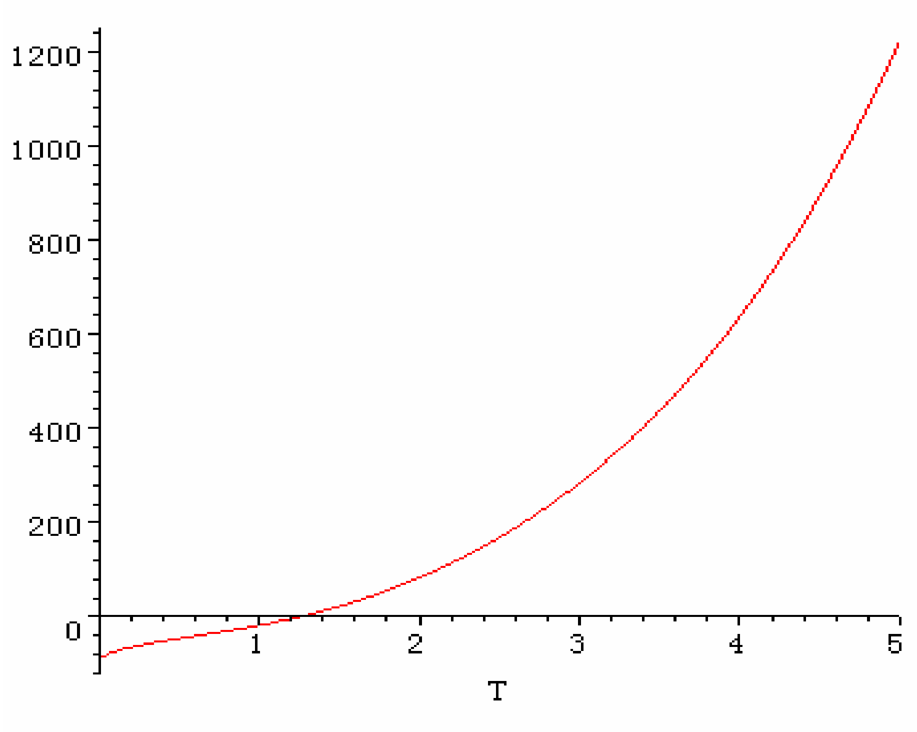

Figure 7. Maturity date and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The figure shows T on the horizontal axis and n^* on the vertical axis.

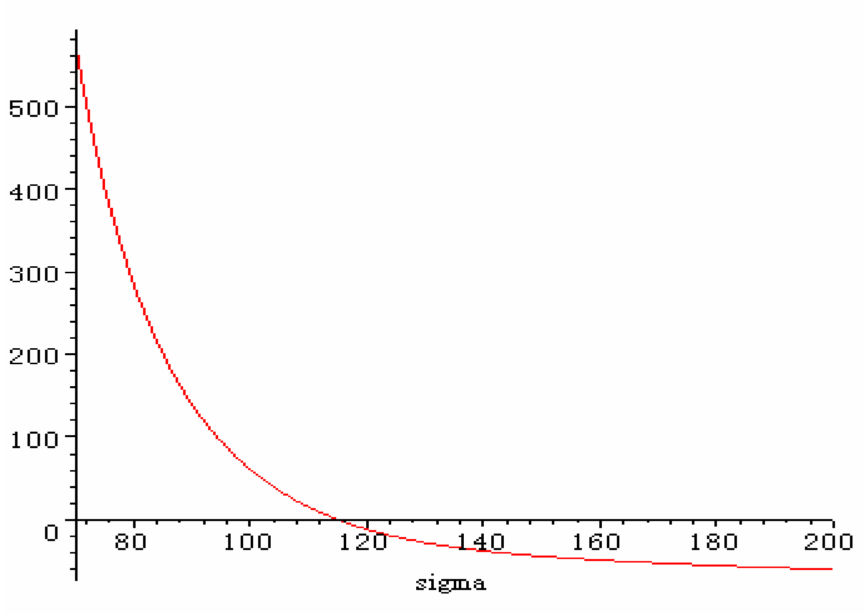

Figure 8. Volatility of profit excluding acquisition-related costs of the merger firm and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The figure shows σ on the horizontal axis and n^* on the vertical axis.

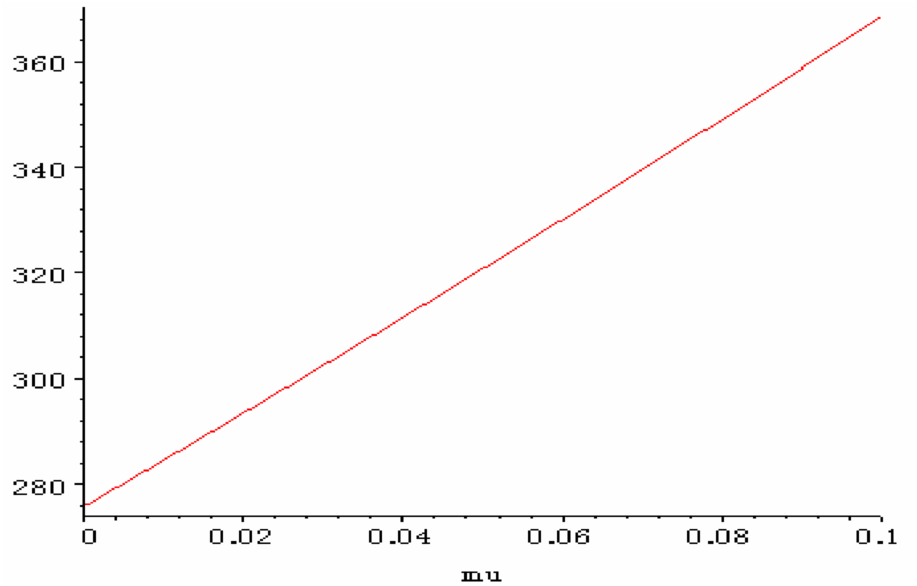

Figure 9. Researchers' drift of successful R&D and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The figure shows $\bar{\mu}$ on the horizontal axis and n^* on the vertical axis.

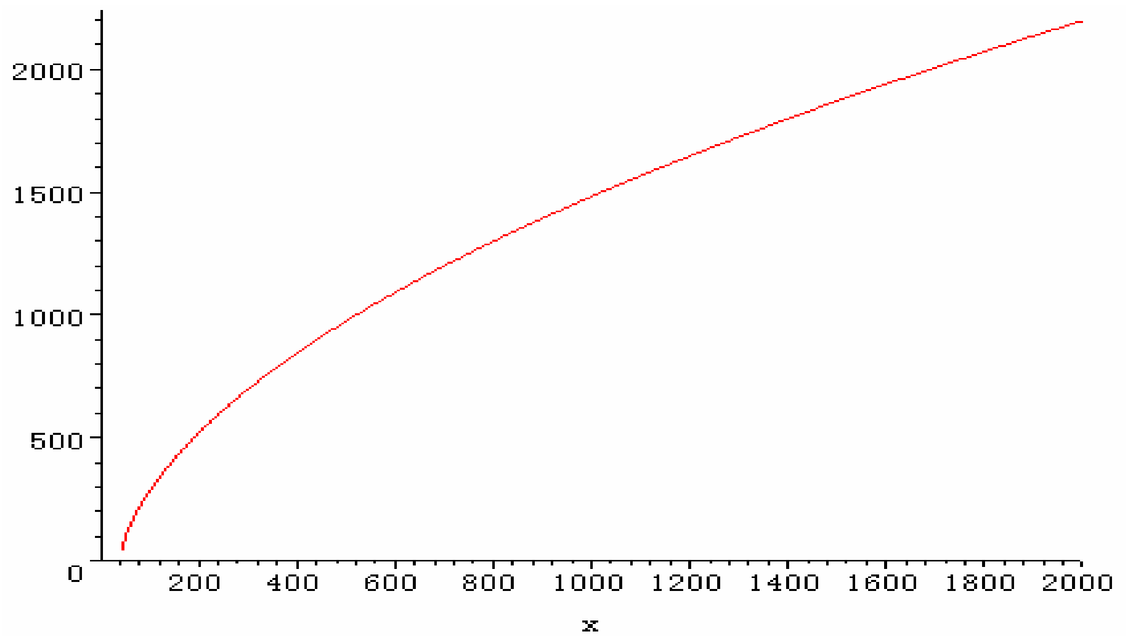

Figure 10. Profit excluding acquisition-related costs of the merger firm and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The figure shows $x(t)$ on the horizontal axis in the range of 0 to 2000 and n^* on the vertical axis.

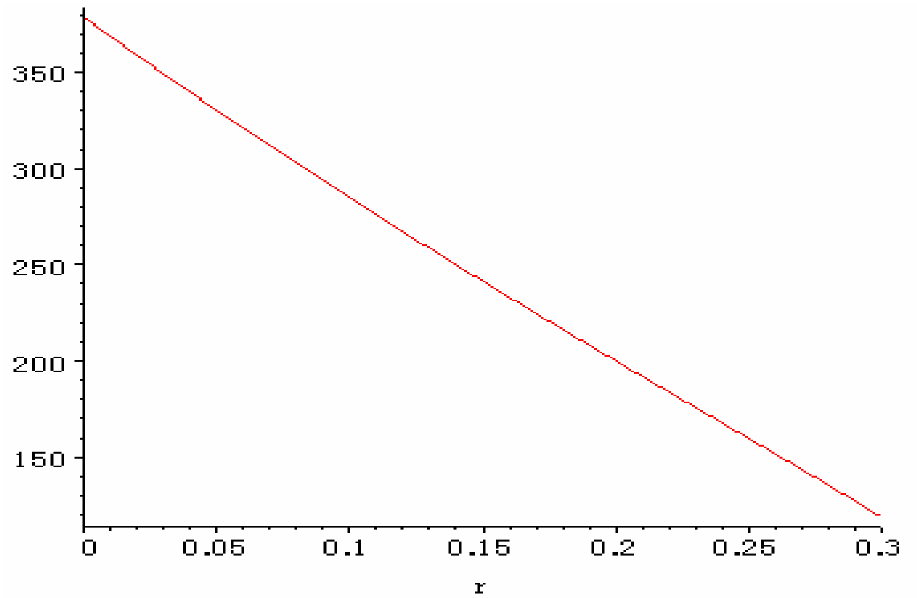

Figure 11. Short-term interest rate and the optimal number of the start-up entrepreneurs to take equity positions in by the merger firm. The figure shows r in the range of 0 to 0.3 on the horizontal axis and n^* on the vertical axis.

III. Concluding Remarks

The profit maximization number of start-up entrepreneurs that the merger firm takes equity positions in trying to acquire innovation is provided and simulated for each case when marketing observe a noise on consumer behavior and when marketing observe no noise. Further, the relationship between the number of start-up entrepreneurs that the merger firm takes equity positions in trying to acquire innovation and profit excluding acquisition-related costs $x(t)$ is proven to be S-shaped with no or with observation noise.

In figure 3, we can see that the merger firm's optimal number of start-up entrepreneurs to take equity positions in varies drastically by the number of firms that are rivals to the merger firm plus the number of the start-up entrepreneurs that the merger firm has no technological interest in. This shows that when its rivals begin to enter the industry, the incumbent (the merger firm) performs more R&D projects and thus takes equity positions in more start-up entrepreneurs to differentiate its products. This is an incentive of "escaping competition" (Aghion et al. 2005). However, when there are excessive rivals in an industry, the incumbent first begins to conduct lesser R&D projects than before, and thus takes equity positions in lesser start-up entrepreneurs and suddenly stop innovating. This is because of the "business stealing" effect (Aghion et al. 2005; Aghion and Howitt 1992; 1998). The implication of this is that there is an optimal number of firms for innovation. In figure 5, we can see that the merger firm's expected profits are shown to be much beyond than those that are not. What are these gains that increased expected profits of the merger firm? Owing to the complete stock markets, the gained assets including intangible and tangible assets (net assets) are arbitrage free. There are no gains or losses from the acquisition itself by the no-arbitrage condition. Note that, in this model, invention by itself does not contribute to profits or sales of the merger firm. If the invented new technology can not suffice the demands of consumers, there are nothing left but the R&D costs. It is the integration of marketing organized for the new technology and the new technology corresponding to the outcome of marketing activities during the period of R&D projects that raised $x(t)$ of the merger firm.

Appendix

Derivation of (7): The variance of Bayesian distribution founded on the normal distributions after updated t times is[8]

$$\frac{\sigma^2\sigma_\theta^2}{\sigma^2+\sigma_\theta^2 t}, \tag{A1}$$

where, σ^2 is the variance of $x(t)$ and σ_θ^2 is the variance of the prior distribution. Assuming $\tilde{m}$ as evenly allocated for each t and i, (A1) will be

$$Var(t)=\frac{\sigma^2\sigma_\theta^2}{\sigma^2+\sigma_\theta^2\tilde{m}t}. \tag{A2}$$

The inverse of this variance, the precision, is

$$\frac{\frac{\sigma^2}{n}+\frac{\sigma_\theta^2}{n}\tilde{m}t}{\frac{\sigma^2}{n}\frac{\sigma_\theta^2}{n}}, \tag{A3}$$

where we replaced σ^2 by $\frac{\sigma^2}{n}$ and σ_θ^2 by $\frac{\sigma_\theta^2}{n}$ to take into consideration of the reductions of the variances by taking equity positions in n start-up entrepreneurs. This can be justified by assuming that the rational merger firm does not let the start-up entrepreneurs that are receiving equity finance from it, have the same prior distributions and undertake the same marketing activities. Differentiating (A3) with respect to t yields

$$\begin{aligned}\{d(\frac{\frac{\sigma^2}{n}+\frac{\sigma_\theta^2}{n}\tilde{m}t}{\frac{\sigma^2}{n}\frac{\sigma_\theta^2}{n}})\}/dt &= \frac{\frac{\sigma_\theta^2}{n}\tilde{m}}{\frac{\sigma^2}{n}\frac{\sigma_\theta^2}{n}} \\ &= \frac{\tilde{m}}{\frac{\sigma^2}{n}} \\ &= \frac{\tilde{m}n}{\sigma^2}. \end{aligned} \tag{A4}$$

We assume that this increment of precision per time as the drift of $x(t)$ through marketing activities. Thus stochastic differential equation describing the incremental process of $x(t)$ is given by (7).

The Proofs of the Existence and Uniqueness of Solutions of (5), (7), and (12): We first provide the proofs of the existence and uniqueness of a solution of (5) in the range of $[0,T]$. The following proof is based on Klebaner (2005, pp.17-18), Minotani (2000), and Øksendal (1998, chap V). (5) satisfies the following condition:

$$|\mu(j,t)| + \left|\sqrt{\frac{j}{n}}(j,t)\right| \leq B(1+|j|); \quad j \in \boldsymbol{R^n}, \ t \in [0,T] \tag{A5}$$

for some constant B, and the following Lipschitz condition:

$$|\mu(j,t) - \mu(l,t)| + \left|\sqrt{\frac{j}{n}}(j,t) - \sqrt{\frac{j}{n}}(l,t)\right| \leq D|j-l|; \quad j,l \in \boldsymbol{R^n}, \ t \in [0,T] \tag{A6}$$

for some constant D. Since μ is the drift of j, $\mu < 1$. Thus, by an adequately large B, (A5) is satisfied, as j is the number of times of success in R&D. We prove the Lipschitz condition below. Here we assume that $\mu(j,t) = \mu_1 j(t) + b$ and $\sqrt{\frac{j}{n}}(j,t) = \sqrt{\frac{1}{n_1}}\sqrt{j(t)} + b_0$. Owing to $\mu < 1$ and an adequately large D, μ_1 and $\sqrt{\frac{1}{n_1}}$ satisfy conditions:

$$|\mu_1| < \frac{D}{2} \text{ and } \sqrt{\frac{1}{n_1}} < \frac{D}{2}. \tag{A7}$$

Therefore,

$$|\mu(j,t) - \mu(l,t)| = |\mu_1\{j(t) - l(t)\}| = |\mu_1||j(t) - l(t)| \leq \frac{D}{2}|j(t) - l(t)|,$$

$$\left|\sqrt{\frac{j}{n}}(j,t) - \sqrt{\frac{l}{n}}(l,t)\right| = \left|\sqrt{\frac{1}{n_1}}\{\sqrt{j(t)} - \sqrt{l(t)}\}\right| = \left|\sqrt{\frac{1}{n_1}}\right|\left|\sqrt{j(t)} - \sqrt{l(t)}\right| \leq \frac{D}{2}|j(t) - l(t)|. \tag{A8}$$

Then the Lipschitz condition is satisfied. Consequently, the existence and uniqueness of a solution of (5) in the range of $[0,T]$ are proved, provided that the second moment of initial value $j(0)$ is

$$E[\{j(0)\}^2] < \infty. \tag{A9}$$

And also $j(0)$ must be independent of $V(t)$ and $t \geq 0$. These conditions are satisfied considering that $j(0)$ is the number of times of success in R&D in time 0 and thus constant. Q.E.D.

We also prove similarly the existence and uniqueness of a solution of (7) in the range of $[0, T]$. (7) satisfies the following condition:

$$\left|\frac{\tilde{m}n}{\sigma^2}(x,t)\right| + \left|\frac{\sigma}{\sqrt{n}}(x,t)\right| \leq K(1+|x|); \quad x \in \boldsymbol{R^n}, \quad t \in [0,T] \tag{A10}$$

for constant K and the Lipschitz condition:

$$\left|\frac{\tilde{m}n}{\sigma^2}(x,t) - \frac{\tilde{m}n}{\sigma^2}(y,t)\right| + \left|\frac{\sigma}{\sqrt{n}}(x,t) - \frac{\sigma}{\sqrt{n}}(y,t)\right| \leq K|x-y|; \quad x, y \in \boldsymbol{R^n}, \quad t \in [0,T]. \tag{A11}$$

If we assume $\sigma^2 > \tilde{m}n$ and seeing that $\tilde{m} \geq 1$ and $n \geq 1$, we get $\frac{\tilde{m}n}{\sigma^2} < 1$. We also assume here that $K = 3\sigma^2$ then, $K > \frac{\sigma}{\sqrt{n}}$. And as $x \geq 1$, (7) satisfies (A10).

We prove the Lipschitz condition below. Here we assume that $\frac{\tilde{m}n}{\sigma^2}(x,t) = \tilde{m}n\frac{1}{\sigma_1^2(x)} + d$ and $\frac{\sigma}{\sqrt{n}}(x,t) = \frac{\sigma_1 \cdot x}{\sqrt{n}} + d_0$. Considering $\sigma^2 > \tilde{m}n$, $\tilde{m} \geq 1$, $n \geq 1$, and $K = 3\sigma^2$, $\frac{1}{\sigma_1^2}$ and $\frac{\sigma_1}{\sqrt{n}}$ satisfy

$$\left|\frac{1}{\sigma_1^2(x)}\right| < x \quad \text{and} \quad \left|\frac{\sigma_1}{\sqrt{n}}\right| < \frac{K}{2}. \tag{A12}$$

Therefore,

$$\left|\frac{\tilde{m}n}{\sigma^2}(x,t) - \frac{\tilde{m}n}{\sigma^2}(y,t)\right| = \left|\tilde{m}n\{\frac{1}{\sigma_1^2(x)} - \frac{1}{\sigma_1^2(y)}\}\right| = |\tilde{m}n|\left|\frac{1}{\sigma_1^2(x)} - \frac{1}{\sigma_1^2(y)}\right| \leq \frac{K}{2}|x-y|,$$

$$\left|\frac{\sigma}{\sqrt{n}}(x,t) - \frac{\sigma}{\sqrt{n}}(y,t)\right| = \left|\frac{\sigma_1}{\sqrt{n}}(x-y)\right| = \left|\frac{\sigma_1}{\sqrt{n}}\right||x-y| \leq \frac{K}{2}|x-y|. \tag{A13}$$

This is because $|\tilde{m}n| \leq \frac{K}{2}$ for $K = 3\sigma^2$. Hence the Lipschitz condition is satisfied. Consequently, the existence and uniqueness of a solution of (7) in the range of $[0,T]$ are proved, provided that the second moment of initial value $x(0)$ is

$$E[\{x(0)\}^2] < \infty. \tag{A14}$$

And also $x(0)$ must be independent of $\tilde{V}(t)$ and $t \geq 0$. These conditions are satisfied for $x(0)$ is the amount of profit excluding acquisition-related costs in time 0 and thus constant. Q.E.D.

The proofs of the existence and uniqueness of a solution of (12) in the range of $[0,T]$ is similarly as above.

Proof of Proposition 1: Because of the Black-Scholes (1973) economy and the constant PER, the call option price is

$$f(t, \bar{j}(t), x(t)) = \log\{\frac{S(t)}{S(0)}\} = \log\{\frac{R(n\frac{\bar{j}(t)}{m}x(t))}{R(x(0))}\}, \tag{A15}$$

where R is the PER. Without loss of generality, we can assume that the number of shares of the merger firm is constant through $[0,T]$. In (A15), since one performs R&D projects and marketing activities in the $[0,T]$, m, $\tilde{m}$, and n are predetermined by the contract made before time 0. Because of complete stock markets, as soon as m, $\tilde{m}$, and n are decided the effects on $S(t)$ are formulated and thus also on $S(0)$ so as to the purchase of assets are arbitrage free. For simplicity, let $r = 0$. Owing to the Black-Scholes (1973) economy, the stock pays no dividends or other distributions thus, in this case, the forward price of $S(t)$ at time 0 equals $S(0)$. Because the expected settlement amount of the forward, the forward price, and the future price coincide (Duffie, 1996, chap. 8, pp.166-168), $n\frac{\bar{j}(t)}{m}x(t)$ and $x(0)$ have the multipliers so that $S(t) = S(0)$ and thus, m, $\tilde{m}$, and n can not affect $f(t, \bar{j}(t), x(t))$. This is nothing but a Radner equilibrium under complete markets. Therefore, there are no effects of m,

$\widetilde{m}$, and n on $f(t, \bar{j}(t), x(t))$. That is,

$$f(t, \bar{j}(t), x(t), m, \widetilde{m}, n) = f(t, \bar{j}(t), x(t))$$

Q.E.D.

LEMMA 1: $E[e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}] = e^{\int_0^t \frac{\bar{j}(s)}{2n} ds}$

Proof of Lemma 1: First we prove that $e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}$ is a martingale. This is since,

$$dZ_0(t) = -Z_0(t)\theta'(t)dW(t)\,, \quad Z_0(0) = 1$$

or equivalently

$$Z_0(t) = 1 - \int_0^t Z_0(s)\theta'(s)dW(s)\,, \quad \forall t \in [0, T]$$

is a local martingale and if $\theta(\cdot)$ is bounded in t and a fixed sample point $\omega \in \Omega$, it is a martingale (Karatzas and Shreve 1998, p.17, Remark 5.2), where in Karatzas and Shreve (1998) $W(t)$ is a Brownian motion, $\omega \in \Omega$, and $(\Omega, \boldsymbol{F}, \boldsymbol{P})$ is a probability space. Expanding the above equation, we have

$$\frac{dZ_0(t)}{Z_0(t)} = -\theta'(t)dW(t) = d\log Z_0(t)\,,$$

integrating the above equation we get

$$\int_0^t d\log Z_0(s) = -\int_0^t \theta'(s)dW(s)$$

$$\log Z_0(t) - \log Z_0(0) = -\int_0^t \theta'(s)dW(s)$$

$$\log Z_0(t) = \log Z_0(0) - \int_0^t \theta'(s) dW(s)$$

$$Z_0(t) = \exp\{-\int_0^t \theta'(s) dW(s)\},$$

where we applied $Z_0(0) = 1$. Thus, $e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}$ is a local martingale. Because $\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} ds$ is bounded in t (since there is the maturity date) and ω (for no researcher can attain successful R&D unboundedly and because of the lower bound g of μ), it is a martingale. Martingale increments are independent to each other because, "*when squaring sums of martingale increments and taking the expectation, one can neglect the cross-product terms* (Karatzas and Shreve (1991, p.32))." Thus, applying $E[e^{\sqrt{\frac{\bar{j}(s)}{n}} V(s)}] = e^{\frac{\bar{j}(s)}{2n} s}$,

$$E[e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}] = E[\exp\{\lim_{l\to\infty} \sum_{h=1}^{l} \sqrt{\frac{\bar{j}(\delta_h)}{n}} (V(s_h) - V(s_{h-1}))\}]$$

$$= E[\lim_{l\to\infty} [\prod_{h=1}^{l} \exp\{\sqrt{\frac{\bar{j}(\delta_h)}{n}} (V(s_h) - V(s_{h-1}))\}]]$$

$$= \lim_{l\to\infty} [\prod_{h=1}^{l} E[\exp\{\sqrt{\frac{\bar{j}(\delta_h)}{n}} (V(s_h) - V(s_{h-1}))\}]]$$

$$= \lim_{l\to\infty} [\prod_{h=1}^{l} \exp\{\frac{\bar{j}(\delta_h)}{2n} (s_h - s_{h-1})\}]$$

$$= \lim_{l\to\infty} [\exp\{\sum_{h=1}^{l} \frac{\bar{j}(\delta_h)}{2n} (s_h - s_{h-1})\}]$$

$$= e^{\int_0^t \frac{\bar{j}(s)}{2n} ds},$$

where $\delta_h \in [s_h, s_{h-1}]$ and $\max(s_l - s_{l-1}) \to 0$. In the above calculations we have used that martingales are independent to each other and thus, the expectations of multiplications are equal to the multiplications of expectations. Q.E.D.

Proof of Proposition 2: Because of the Black-Scholes (1973) economy and the constant PER, the call option price is

$$f(t,\bar{j}(t),x(t)) = \log\{\frac{S(t)}{S(0)}\} = \log\{\frac{R(n\frac{\bar{j}(t)}{m}x(t))}{R(x(0))}\}. \qquad \text{(A15)}$$

In (A15), $nc + C$ are not subtracted from $n\frac{\bar{j}(t)}{m}x(t)$, because there are no gains or losses from the acquisition itself by the no-arbitrage condition so that we can focus our study on the profits resulted from innovation and not from the arbitrage of the acquisition. The gained assets including intangible and tangible (net assets) are arbitrage free owing to the complete stock markets of the financial sector. Thus, acquisition-related costs have no influence on $S(t)$. Therefore, $nc + C$ are not subtracted from (A15). Hence, though $S(t)$ is a function of price/book-value ratio (PBR), PBR is not included in (A15) as PER is.

For simplicity of calculations, first we let $r = 0$ and afterward we take into consideration of the constant short-term interest rate r.

By Ito's formula, (5), (7), $m = \tilde{m}$, and proposition 1, (A15) can be shown as the following stochastic process[9]

$$\begin{aligned}
df(t,\bar{j}(t),x(t)) &= [f_1 + \frac{1}{2}\{f_{22}(\bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}})^2 + f_{33}(x(t)\frac{\sigma}{\sqrt{n}})^2\} + 2f_{23}\bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}}x(t)\frac{\sigma}{\sqrt{n}}dV(t)d\tilde{V}(t)]dt \\
&\quad + \frac{1}{\bar{j}(t)}\{\bar{\mu}\bar{j}(t)dt + \bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}}dV(t)\} + \frac{1}{x(t)}\{\frac{mn}{\sigma^2}x(t)dt + x(t)\frac{\sigma}{\sqrt{n}}d\tilde{V}(t)\} \\
&= 0 - \frac{\bar{j}(t)}{2n}dt - \frac{\sigma^2}{2n}dt + \bar{\mu}dt + \sqrt{\frac{\bar{j}(t)}{n}}dV(t) + \frac{mn}{\sigma^2}dt + \frac{\sigma}{\sqrt{n}}d\tilde{V}(t) \\
&= -\frac{\bar{j}(t)}{2n}dt - \frac{\sigma^2}{2n}dt + \bar{\mu}dt + \frac{mn}{\sigma^2}dt + \sqrt{\frac{\bar{j}(t)}{n}}dV(t) + \frac{\sigma}{\sqrt{n}}d\tilde{V}(t), \qquad \text{(A16)}
\end{aligned}$$

where the subscripts of f denote the partial derivatives. From the above equation,

$$f(t,\bar{j}(t),x(t)) = \int_0^t d\,f(t,\bar{j}(t),x(t))$$

$$= \int_0^t (\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\bar{j}(s)}{2n} - \frac{\sigma^2}{2n})ds + \int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s) + \int_0^t \frac{\sigma}{\sqrt{n}} d\tilde{V}(s).$$

From (A15),

$$\log\{n\frac{\bar{j}(t)}{m}x(t)\} = \log x(0) + \int_0^t (\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\bar{j}(s)}{2n} - \frac{\sigma^2}{2n})ds + \int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s) + \int_0^t \frac{\sigma}{\sqrt{n}} d\tilde{V}(s)$$

$$n\frac{\bar{j}(t)}{m}x(t) = x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\sigma^2}{2n})t - \int_0^t \frac{\bar{j}(s)}{2n}ds + \int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s) + \frac{\sigma}{\sqrt{n}}\tilde{V}(t)\}$$

(Here, we are using $\int_0^t dV(s) = V(t) - V(0) = V(t)$ since $V(0) = 0$. See Karatzas and Shreve (1991, p.47) for $V(0) = 0$.)

$$= x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\sigma^2}{2n})t - \int_0^t \frac{\bar{j}(s)}{2n}ds\} e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)} e^{\frac{\sigma}{\sqrt{n}}\tilde{V}(t)}.$$

Taking expectation on the both sides gives

$$E[n\frac{\bar{j}(t)}{m}x(t)] = x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\sigma^2}{2n})t - \int_0^t \frac{\bar{j}(s)}{2n}ds\} e^{\int_0^t \frac{\bar{j}(s)}{2n}ds} e^{\frac{\sigma^2}{2n}t}.$$

This is because of $E[e^{\sigma w}] = e^{\frac{\sigma^2}{2}t}$, $E[e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}] = e^{\int_0^t \frac{\bar{j}(s)}{2n}ds}$ and $E[e^{\frac{\sigma}{\sqrt{n}}\tilde{V}(t)}] = e^{\frac{\sigma^2}{2n}t}$, where w is the Brownian motion. See lemma 1 in the Appendix for $E[e^{\int_0^t \sqrt{\frac{\bar{j}(s)}{n}} dV(s)}] = e^{\int_0^t \frac{\bar{j}(s)}{2n}ds}$.
Thus,

$$E[n\frac{\bar{j}(t)}{m}x(t)] = x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - \frac{\sigma^2}{2n} + \frac{\sigma^2}{2n})t - \int_0^t \frac{\bar{j}(s)}{2n}ds + \int_0^t \frac{\bar{j}(s)}{2n}ds\}$$

$$= x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2})t\}. \tag{A17}$$

The above equation is the expected production function of the merger firm by taking equity positions in n start-up entrepreneurs. Hereafter, we investigate the characteristics of this production function with respect to n.

The expected average product AP_n and the expected marginal product MP_n of (A17) is

$$\frac{1}{n}E[n\frac{\bar{j}(t)}{m}x(t)] = \frac{1}{n}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\} \tag{A18}$$

$$\frac{\partial E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n} = \frac{\partial}{\partial n}[x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}]$$

$$= \frac{mk}{\sigma^2 N}tx(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}, \tag{A19}$$

where we are using

$$\frac{\partial \frac{mn}{\sigma^2}}{\partial n} = \frac{\partial}{\partial n}\frac{\frac{M}{n+k}n}{\sigma^2}$$

$$= \frac{m}{\sigma^2} + \frac{n}{\sigma^2}\frac{\partial \frac{M}{n+k}}{\partial(n+k)}\frac{\partial(n+k)}{\partial n}$$

$$= \frac{m}{\sigma^2} + \frac{n}{\sigma^2}\{\frac{-M}{(n+k)^2}\}$$

$$= \frac{m}{\sigma^2} - \frac{n}{\sigma^2}\frac{m}{(n+k)}$$

$$= \frac{m}{\sigma^2}(1-\frac{n}{n+k})$$

$$= \frac{m}{\sigma^2}\frac{k}{n+k}$$

$$= \frac{m}{\sigma^2}\frac{k}{N}.$$

And the second order derivative of (A17) becomes

$$\frac{\partial^2 E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n^2} = \frac{\partial}{\partial n}[\frac{mk}{\sigma^2 N}t]x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\} + (\frac{mk}{\sigma^2 N}t)^2 x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}$$

$$= \{(\frac{mk}{\sigma^2 N}t)^2 - 2\frac{mk}{\sigma^2 N^2}t\}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}$$

$$= \{\frac{m^2k^2}{\sigma^4 N^2}t^2 - 2\frac{mk}{\sigma^2 N^2}t\}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}$$

$$= \{\frac{mkt}{\sigma^2 N^2}(\frac{mkt}{\sigma^2}-2)\}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}\,, \qquad \text{(A20)}$$

where in the above equation, we are adopting

$$\frac{\partial}{\partial n}[\frac{mk}{\sigma^2 N}t] = \frac{\partial}{\partial n}[\frac{Mkt}{\sigma^2 N^2}]$$

$$= \frac{Mkt}{\sigma^2}\frac{\partial\frac{1}{(n+k)^2}}{\partial(n+k)}\frac{\partial(n+k)}{\partial n}$$

$$= -2\frac{Mkt}{\sigma^2 N^3}$$

$$= -2\frac{mkt}{\sigma^2 N^2}\,.$$

And putting $\frac{mkt}{\sigma^2}-2$ in (A20) equal to naught, we get

$$(\frac{mkt}{\sigma^2}-2) = 0$$

$$mkt = 2\sigma^2$$

$$\frac{M}{n+k}kt = 2\sigma^2$$

$$Mkt = 2\sigma^2(n+k)$$

$$Mkt - 2\sigma^2 k = 2\sigma^2 n$$

$$n = \frac{Mkt}{2\sigma^2} - k\,.$$

Therefore,

$$n < \frac{Mkt}{2\sigma^2} - k \;\Leftrightarrow\; \frac{\partial^2 E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n^2} > 0$$

$$n = \frac{Mkt}{2\sigma^2} - k \;\Leftrightarrow\; \frac{\partial^2 E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n^2} = 0$$

$$n > \frac{Mkt}{2\sigma^2} - k \;\Leftrightarrow\; \frac{\partial^2 E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n^2} < 0. \qquad \text{(A21)}$$

Hence in the range of $0 < n < \frac{Mkt}{2\sigma^2} - k$, MP_n increases, is at the maximum at $n = \frac{Mkt}{2\sigma^2} - k$, and decreases in the range of $n > \frac{Mkt}{2\sigma^2} - k$. And differentiating AP_n with respect to n yields

$$\frac{\partial}{\partial n}[\frac{1}{n}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}] = -\frac{1}{n^2}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\} + \frac{1}{n}\frac{mkt}{\sigma^2 N}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}$$

$$= x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2})t\}\frac{1}{n}(\frac{mkt}{\sigma^2 N} - \frac{1}{n}). \qquad \text{(A22)}$$

And putting the last parenthesis in the above equation equal to zero, we have

$$(\frac{mkt}{\sigma^2 N} - \frac{1}{n}) = \frac{Mkt}{\sigma^2 (n+k)^2} - \frac{1}{n} = 0,$$

and then investigating the maximum point of AP_n with respect to n brings

$$Mktn = (n^2 + 2nk + k^2)\sigma^2$$

$$n^2\sigma^2 + 2nk\sigma^2 + k^2\sigma^2 - Mktn = 0$$

$$n^2\sigma^2 + n(2k\sigma^2 - Mkt) + k^2\sigma^2 = 0$$

$$n = \frac{-(2k\sigma^2 - Mkt) \pm \sqrt{(2k\sigma^2 - Mkt)^2 - 4\sigma^2 k^2 \sigma^2}}{2\sigma^2}$$

$$= \frac{-(2k\sigma^2 - Mkt) \pm \sqrt{4k^2\sigma^4 - 4k^2\sigma^2 Mt + M^2k^2t^2 - 4\sigma^4 k^2}}{2\sigma^2}$$

$$= \frac{-2k\sigma^2 + Mkt \pm \sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$$

$$= \frac{Mkt}{2\sigma^2} - k + \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}. \qquad \text{(A23)}$$

In the above equation, it is obvious from (A21) to be

$n = \frac{Mkt}{2\sigma^2} - k + \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$ and not $n = \frac{Mkt}{2\sigma^2} - k - \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$.

From (A23), AP_n increases in the range of $0 < n < \frac{Mkt}{2\sigma^2} - k + \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$, is at the maximum when $n = \frac{Mkt}{2\sigma^2} - k + \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$, and decreases in the range of $n > \frac{Mkt}{2\sigma^2} - k + \frac{\sqrt{-4k^2\sigma^2 Mt + M^2k^2t^2}}{2\sigma^2}$. If $Mt > 4\sigma^2$, then the relationship between n and $x(t)$ is S-shape.

As the expectation of an expectation is the expectation,

$$\pi = E[\pi] = E[n\frac{\bar{j}(t)}{m}x(t)] - nc - C. \tag{A24}$$

The first order condition of (A24) is

$$\begin{aligned}
\frac{\partial \pi}{\partial n} &= \frac{\partial}{\partial n}[x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - r)t\} - nc - C] \\
&= \frac{mkt}{\sigma^2 N}x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - r)t\} - c \\
&= \frac{Mkt}{\sigma^2 N^2}x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2} - r)t\} - c \\
&= \frac{Mkt}{\sigma^2(n+k)^2}x(0)\exp\{(\bar{\mu} + \frac{Mn}{\sigma^2(n+k)} - r)t\} - c = 0,
\end{aligned} \tag{A25}$$

where we discounted $x(t)$ by r. Solving the above equation for n when $t = T$ by Maple gives

$$n^* = -k - \frac{MkT}{2\sigma^2 W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2 + M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}})}. \tag{10}$$

If $-1 < W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}}) < 0$, then n^* is the profit maximization point. If

$-\frac{1}{e} < -\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}} < 0$, then there are 2 solutions for (A25) and since

$-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}}$ is a real number, it becomes $-\frac{1}{e} < -\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}} < 0$.[10] As the relationship between n and $x(t)$ is S-shape, one solution is the profit maximization point and the other is the profit minimization point. When

$-1 < W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}}) < 0$, the solution is the profit maximization point and when

$W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)\sigma^2+M\}}{2\sigma^2}T]}}{\sigma^2\sqrt{\frac{x(0)}{Mc\sigma^2 kT}}}) < -1$, the solution is the profit minimization point.

Derivation of (11): (7) is a stochastic differential equation when the characteristics of consumers are perfectly known. When consumers' characteristics are not perfectly known or marketing observe a noise on consumer behavior, the precision about the true value of the four Ps that brings the largest $x(t)$ is

$$\rho_1 = \rho + \frac{1}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2}, \tag{A26}$$

where ρ is the inverse of variance of the prior distribution σ_θ^2 and σ_η^2 is the variance of an observation noise (Chamley (2004, pp.48-50)). Updating (A26), t times similarly when marketing observe no noise yields

$$\rho_t = \rho + \frac{t}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2}. \tag{A27}$$

Assuming that $\tilde{m}$ as evenly allocated for each unit of time and firm, and replacing σ^2 by $\frac{\sigma^2}{n}$, σ_η^2 by $\frac{\sigma_\eta^2}{n}$, and $\rho = \frac{1}{\sigma_\theta^2}$ by $\frac{n}{\sigma_\theta^2}$ to consider the reductions of the variances and the augment of precision by the merger firm taking equity positions in n start-up entrepreneurs, we get

$$\rho_t = \rho + \frac{t\tilde{m}n}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2}. \tag{A28}$$

Differentiating the above equation with respect to t gives

$$\frac{\partial \rho_t}{\partial t} = \frac{\tilde{m}n}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2}. \tag{A29}$$

Assume that this increment of the inverse of the Bayesian inference founded on the normal distributions per time with an observation noise to be the drift of $x(t)$. Then we can remodel the incremental process of $x(t)$ in (7) as (11).

Proof of Proposition 3: Because of the Black-Scholes (1973) economy and the constant PER, the call option price is

$$f(t, \bar{j}(t), x(t)) = \log\{\frac{S(t)}{S(0)}\} = \log\{\frac{R(n\frac{\bar{j}(t)}{m}x(t))}{R(x(0))}\}. \tag{A15'}$$

By Ito's formula, (5), (11), $m = \tilde{m}$, and proposition 1, (A15)'can be shown as the following stochastic process

$$df(t,\bar{j}(t),x(t)) = [f_1 + \frac{1}{2}\{f_{22}(\bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}})^2 + f_{33}(x(t)\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}})^2\} + 2f_{23}\bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}}x(t)\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}dV(t)d\tilde{V}(t)]dt$$

$$+\frac{1}{\bar{j}(t)}\{\bar{\mu}\bar{j}(t)dt + \bar{j}(t)\sqrt{\frac{\bar{j}(t)}{n}}dV(t)\} + \frac{1}{x(t)}\{\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}x(t)dt + x(t)\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}d\tilde{V}(t)\}$$

$$= 0 - \frac{\bar{j}(t)}{2n}dt - \frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n}dt + \bar{\mu}dt + \sqrt{\frac{\bar{j}(t)}{n}}dV(t) + \frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}dt + \sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}d\tilde{V}(t)$$

$$=-\frac{\bar{j}(t)}{2n}dt-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n}dt+\bar{\mu}dt+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}dt+\sqrt{\frac{\bar{j}(t)}{n}}dV(t)+\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}d\tilde{V}(t). \qquad \text{(A16)'}$$

From the above equation,

$$f(t,\bar{j}(t),x(t))=\int_0^t d\,f(t,\bar{j}(t),x(t))$$

$$=\int_0^t(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\bar{j}(s)}{2n}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})ds+\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)+\int_0^t\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}d\tilde{V}(s).$$

From (A15)',

$$\log\{n\frac{\bar{j}(t)}{m}x(t)\}=\log x(0)+\int_0^t(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\bar{j}(s)}{2n}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})ds+\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)+\int_0^t\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}d\tilde{V}(s)$$

$$n\frac{\bar{j}(t)}{m}x(t)=x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})t-\int_0^t\frac{\bar{j}(s)}{2n}ds+\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)+\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}\tilde{V}(t)\}$$

(Here, we are using $\int_0^t dV(s)=V(t)-V(0)=V(t)$ since $V(0)=0$. See Karatzas and Shreve (1991, p.47) for $V(0)=0$.)

$$=x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})t-\int_0^t\frac{\bar{j}(s)}{2n}ds\}e^{\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)}e^{\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}\tilde{V}(t)}.$$

Taking expectation on the both sides gives

$$E[n\frac{\bar{j}(t)}{m}x(t)]=x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})t-\int_0^t\frac{\bar{j}(s)}{2n}ds\}e^{\int_0^t\frac{\bar{j}(s)}{2n}ds}e^{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n}t}.$$

This is because of $E[e^{\sigma w}]=e^{\frac{\sigma^2}{2}t}$, $E[e^{\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)}]=e^{\int_0^t\frac{\bar{j}(s)}{2n}ds}$ and

$E[e^{\sqrt{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{n}}\tilde{V}(t)}]=e^{\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n}t}$. See lemma 1 for $E[e^{\int_0^t\sqrt{\frac{\bar{j}(s)}{n}}dV(s)}]=e^{\int_0^t\frac{\bar{j}(s)}{2n}ds}$. Thus,

$$E[n\frac{\bar{j}(t)}{m}x(t)]=x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}-\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n}+\frac{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}{2n})t-\int_0^t\frac{\bar{j}(s)}{2n}ds+\int_0^t\frac{\bar{j}(s)}{2n}ds\}$$

$$=x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2})t\}. \quad \text{(A17)'}$$

The above equation is the expected production function of the merger firm by taking equity positions in n start-up entrepreneurs when there is an observation noise on consumer behavior. Hereafter, we investigate the characteristics of this production function with respect to n.

The expected average product $AP_n^{'}$ and the expected marginal product $MP_n^{'}$ of (A17)' is

$$\frac{1}{n}E[n\frac{\bar{j}(t)}{m}x(t)]=\frac{1}{n}x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2})t\} \quad \text{(A18)'}$$

$$\frac{\partial E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n}=\frac{\partial}{\partial n}[x(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2})t\}]$$

$$=\frac{mk}{\{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2\}N}tx(0)\exp\{(\bar{\mu}+\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2})t\}, \quad \text{(A19)'}$$

where we are using

$$\frac{\partial\frac{mn}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}}{\partial n}=\frac{\partial}{\partial n}\frac{\frac{M}{n+k}n}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}$$

$$=\frac{m}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}+\frac{n}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}\frac{\partial\frac{M}{n+k}}{\partial(n+k)}\frac{\partial(n+k)}{\partial n}$$

$$=\frac{m}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}+\frac{n}{\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2}\{\frac{-M}{(n+k)^2}\}$$

$$= \frac{m}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} - \frac{n}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} \frac{m}{(n+k)}$$

$$= \frac{m}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} (1 - \frac{n}{n+k})$$

$$= \frac{m}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} \frac{k}{n+k}$$

$$= \frac{m}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} \frac{k}{N} .$$

And the second order derivative of (A17)' becomes

$$\frac{\partial^2 E[n\frac{\bar{j}(t)}{m}x(t)]}{\partial n^2} = \frac{\partial}{\partial n}[\frac{mk}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2 N}t]x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2})t\} + (\frac{mk}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2 N}t)^2 x(0)\exp\{(\bar{\mu} + \frac{mn}{\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2})t\}$$

$$= \{(\frac{mk}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2 N} t)^2 - 2\frac{mk}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2 N^2} t\} x(0) \exp\{(\bar{\mu} + \frac{mn}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2})t\}$$

$$= \{\frac{m^2k^2}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]^2 N^2} t^2 - 2\frac{mk}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N^2} t\} x(0) \exp\{(\bar{\mu} + \frac{mn}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2})t\}$$

$$= \{\frac{mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N^2} (\frac{mkt}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} - 2)\} x(0) \exp\{(\bar{\mu} + \frac{mn}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2})t\} , \text{(A20)'}$$

where in the above equation, we are adopting

$$\frac{\partial}{\partial n}[\frac{mk}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N} t] = \frac{\partial}{\partial n}[\frac{Mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N^2}]$$

$$= \frac{Mkt}{\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2} \frac{\partial \frac{1}{(n+k)^2}}{\partial (n+k)} \frac{\partial (n+k)}{\partial n}$$

$$= -2\frac{Mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N^3}$$

$$= -2\frac{mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2] N^2} .$$

And putting $\frac{mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - 2$ in (A20)' equal to naught, we get

$$(\frac{mkt}{[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - 2) = 0$$

$$mkt = 2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]$$

$$\frac{M}{n+k} kt = 2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]$$

$$Mkt = 2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2](n+k)$$

$$Mkt - 2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]k = 2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]n$$

$$n = \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k\,.$$

Therefore,

$$n < \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k \quad \Leftrightarrow \quad \frac{\partial^2 E[n \frac{\bar{j}(t)}{m} x(t)]}{\partial n^2} > 0$$

$$n = \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k \quad \Leftrightarrow \quad \frac{\partial^2 E[n \frac{\bar{j}(t)}{m} x(t)]}{\partial n^2} = 0$$

$$n > \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k \quad \Leftrightarrow \quad \frac{\partial^2 E[n \frac{\bar{j}(t)}{m} x(t)]}{\partial n^2} < 0\,. \qquad \text{(A21)'}$$

Hence in the range of $0 < n < \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k$, $MP_n^{'}$ increases, is at the maximum at $n = \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k$, and decreases in the range of $n > \frac{Mkt}{2[\sigma^2 + \sigma_\eta^2 (1+\rho\sigma^2)^2]} - k$. And differentiating $AP_n^{'}$ with respect to n yields

$$\frac{\partial}{\partial n}\left[\frac{1}{n} x(0)\exp\{(\bar{\mu} + \frac{mn}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]})t\}\right] = -\frac{1}{n^2} x(0)\exp\{(\bar{\mu} + \frac{mn}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]})t\} + \frac{1}{n}\frac{mkt}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]N} x(0)\exp\{(\bar{\mu} + \frac{mn}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]})t\}$$

$$= x(0)\exp\{(\bar{\mu} + \frac{mn}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]})t\}\frac{1}{n}\left(\frac{mkt}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]N} - \frac{1}{n}\right). \qquad \text{(A22)'}$$

And putting $\frac{mkt}{[\sigma^2 + \sigma_\eta^2(1+\rho\sigma^2)^2]N} - \frac{1}{n}$ in the above equation equal to zero, we have

$$\left(\frac{mkt}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]N}-\frac{1}{n}\right)=\frac{Mkt}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2](n+k)^2}-\frac{1}{n}=0,$$

and then investigating the maximum point of $AP_n^{'}$ with respect to n brings

$$Mktn=(n^2+2nk+k^2)[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]$$

$$n^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]+2nk[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]+k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]-Mktn=0$$

$$n^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]+n(2k[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]-Mkt)+k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]=0$$

$$n=\frac{-(2k[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]-Mkt)\pm\sqrt{(2k[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]-Mkt)^2-4[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]^2k^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$$

$$=\frac{-(2k[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]-Mkt)\pm\sqrt{4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]^2-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2-4[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]^2k^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$$

$$=\frac{-2k[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]+Mkt\pm\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$$

$$=\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k+\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}. \qquad \text{(A23)'}$$

In the above equation, it is obvious from (A21)' to be

$$n=\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k+\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$$ and not

$$n=\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k-\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$$. From (A23)',

$AP_n^{'}$ increases in the range of

$$0<\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k+\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}<n,$$ is at the maximum

when $n=\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k+\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$, and

decreases in the range of $n>\frac{Mkt}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-k+\frac{\sqrt{-4k^2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]Mt+M^2k^2t^2}}{2[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}$.

If $Mt>4[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]$, then the relationship between n and $x(t)$ is S－shape. As the expectation of an expectation is the expectation,

$$\pi=E[\pi]=E[n\frac{\bar{j}(t)}{m}x(t)]-nc-C. \qquad \text{(A24)'}$$

The first order condition of (A24)' is

$$\begin{aligned}\frac{\partial\pi}{\partial n}&=\frac{\partial}{\partial n}[x(0)\exp\{(\bar{\mu}+\frac{mn}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-r)t\}-nc-C]\\&=\frac{mkt}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]N}x(0)\exp\{(\bar{\mu}+\frac{mn}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-r)t\}-c\\&=\frac{Mkt}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]N^2}x(0)\exp\{(\bar{\mu}+\frac{mn}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2]}-r)t\}-c\\&=\frac{Mkt}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2](n+k)^2}x(0)\exp\{(\bar{\mu}+\frac{Mn}{[\sigma^2+\sigma_\eta^2(1+\rho\sigma^2)^2](n+k)}-r)t\}-c=0,\end{aligned} \qquad \text{(A25)'}$$

where we discounted $x(t)$ by r. Solving the above equation for n when $t=T$ by Maple gives

$$n^*=-k-\frac{MkT\sigma_\theta^4}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)W\left(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}}\right)}. \qquad (12)$$

If $-1 < W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}}) < 0$, then n^* is the profit maximization point. If

$$-\frac{1}{e} < -\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}} < 0$$, then there are 2 solutions for (A25)' and since $-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}}$ is a real number, it becomes $-\frac{1}{e} < -\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}} < 0$. As the relationship between n and $x(t)$ is S-shape, one solution is the profit maximization point and the other is the profit minimization point. When

$$-1 < W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}}) < 0$$, the solution is the profit maximization point and when

$$W(-\frac{1}{2}\frac{e^{[-\frac{\{(\bar{\mu}-r)(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)+M\sigma_\theta^4\}}{2(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^4\sigma^2+\sigma_\eta^2\sigma^4)}T]}}{\sqrt{\frac{x(0)}{Mc\sigma_\theta^4kT}(\sigma^2\sigma_\theta^4+\sigma_\eta^2\sigma_\theta^4+2\sigma_\eta^2\sigma_\theta^2\sigma^2+\sigma_\eta^2\sigma^4)}}) < -1$$, the solution is the profit minimization point.

Derivation of (A26): When a private signal of consumer behavior obtained by marketing of firm i is

$$v_{it} = \theta_{it} + \varepsilon_{it}, \tag{A30}$$

where t indicates time t, θ_{it} is the true value of consumer behavior but unknown, and ε_{it} is an error that is independent of θ_{it} and normally distributed: $n(0,\sigma^2)$, where the mean of ε_{it} is 0 and the variance of it is σ^2. θ_{it} is unknown, but it has the subjective distribution. Random variable v_{it} is normally distributed $n(\theta_{it},\sigma^2)$ and the subjective distribution of θ_{it} belongs to $n(a,\sigma_\theta^2)$. Then, because of the Bayes' rule, the Bayesian equation founded on the normal distribution has a random distribution[8]

$$f(\theta_{it}|v_{it}) \sim n(\frac{\rho a+\rho_\varepsilon v_{it}}{\rho+\rho_\varepsilon},\frac{1}{\rho+\rho_\varepsilon}), \qquad \text{(A31)}$$

where $\rho_\varepsilon = \frac{1}{\sigma^2}$. Let $\rho+\rho_\varepsilon=\rho'$. Since probability is updated by the subjective prior,

$$\rho' = \rho + \rho_\varepsilon \qquad \text{(A32)}$$

and

$$\rho'' = \rho' + \rho_\varepsilon . \qquad \text{(A33)}$$

Substituting (A32) into (A33) yields

$$\begin{aligned}\rho'' &= \rho + \rho_\varepsilon + \rho_\varepsilon \\ &= \rho + 2\rho_\varepsilon .\end{aligned} \qquad \text{(A34)}$$

Thus updating t times brings

$$\rho^{(t)} = \rho + t\rho_\varepsilon . \qquad \text{(A35)}$$

Thus,

$$\begin{aligned}Var_t &= \frac{1}{\rho+t\rho_\varepsilon} \\ &= \frac{1}{\frac{1}{\sigma_\theta^2}+\frac{t}{\sigma^2}} \\ &= \frac{1}{\frac{t\sigma_\theta^2+\sigma^2}{\sigma_\theta^2\sigma^2}}\end{aligned}$$

$$= \frac{\sigma^2 \sigma_\theta^2}{\sigma^2 + t\sigma_\theta^2}, \qquad \text{(A1)}$$

where Var_t is the variance of the Bayesian inference after updated t times. We prove the following based on Chamley (2004, pp.48-50). From (A31), the mean of posterior distribution is

$$\frac{\rho a + \rho_\varepsilon v_{it}}{\rho + \rho_\varepsilon} = \frac{\rho_\varepsilon v_{it}}{\rho + \rho_\varepsilon} + \frac{\rho a}{\rho + \rho_\varepsilon}$$
$$= \alpha v_{it} + (1-\alpha)a,$$

where $\alpha = \frac{\rho_\varepsilon}{\rho + \rho_\varepsilon} = \frac{\rho_\varepsilon}{\rho'}$. When marketing observe a noise or equivalently imperfect information on characteristics of consumer behavior, an observation of consumer behavior b_t by one marketing activity is

$$b_t = \alpha_t v_{it} + (1-\alpha_t)a + \eta_t$$
$$= \alpha_t \theta_{it} + (1-\alpha_t)a + \alpha_t \varepsilon_{it} + \eta_t, \qquad \text{(A36)}$$

where η_t is an observation noise and $\alpha_t \varepsilon_{it} + \eta_t$ is a noise term. η_t and ε_{it} are independent of each other. Transforming the above equation brings

$$\frac{b_t - (1-\alpha_t)a}{\alpha_t} = z_t = \theta_{it} + \varepsilon_{it} + \frac{\eta_t}{\alpha_t}. \qquad \text{(A37)}$$

The variable b_t is *informationally equivalent* to the variables $z_t \sim n(\theta_{it}, \sigma^2 + \frac{\sigma_\eta^2}{\alpha_t^2})$, where σ_η^2 is the variance of η. Therefore, when marketing observe a noise on consumer behavior, (A32) becomes

$$\rho' = \rho + \frac{1}{\sigma^2 + \frac{\sigma_\eta^2}{\alpha_t^2}}$$
$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 (\frac{\rho + \rho_\varepsilon}{\rho_\varepsilon})^2}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \left(\frac{\frac{1}{\sigma_\theta^2} + \frac{1}{\sigma^2}}{\frac{1}{\sigma^2}} \right)^2}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \left(\frac{\frac{\sigma_\theta^2 + \sigma^2}{\sigma_\theta^2 \sigma^2}}{\frac{1}{\sigma^2}} \right)^2}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \left(\frac{\sigma_\theta^2 + \sigma^2}{\sigma_\theta^2} \right)^2}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \frac{\sigma_\theta^4 + 2\sigma_\theta^2 \sigma^2 + \sigma^4}{\sigma_\theta^4}}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \left(1 + \frac{2\sigma^2}{\sigma_\theta^2} + \frac{\sigma^4}{\sigma_\theta^4} \right)}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 \left(1 + \frac{\sigma^2}{\sigma_\theta^2} \right)^2}$$

$$= \rho + \frac{1}{\sigma^2 + \sigma_\eta^2 (1 + \rho\sigma^2)^2} . \qquad \text{(A26)}$$

Using the same technique as in deriving (A35) gives us (A27).

Notes:

[1] Pratt, Raiffa, and Schlaifer (1995) derive (1) by assuming that the prior distribution is the gamma-1 distribution. Definition of the function is given in Pratt, Raiffa, and Schlaifer (1995, p. 202). See Loredo (1990) and Rainwater and Wu (1947) for the theoretical frameworks that derive exactly the same equation as the gamma-1 distribution. Gregory (2005, pp.376-378) derives an application of it based on Loredo (1992) with some theoretical expansion. Both Loredo (1990, 1992) and Gregory (2005) use the Jeffrey's prior as the prior distribution for deriving it. See Jaynes (2003) for the general discussion of the Jeffrey's prior.

[2] See the Appendix for the proofs of the existence and uniqueness of a solution of (5).

[3] See the Appendix. Also see the Appendix for the proofs of the existence and uniqueness of a solution of (7).

[4] See Aghion and Howitt (1998, p.55) for $n\frac{\bar{j}(t)}{m}$.

[5] Use mathematical software such as Maple for this derivation.

[6] See Corless et al. (1996, pp.330-331 and Figure 1) for the *Lambert W function*.

[7] See the derivation of (11) in the Appendix.

[8] See Chamley (2004, p.25) for example.

[9] See Elliott and Kopp (1999, p.124) for example.

[10] See Corless et al. (1996, pp.330-331 and Figure 1)

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